

# On Canonical Homomorphisms of Tensor Sheaves

Jianke Chen\*

## Abstract

In this paper we define tensor modules(sheaves) of *Schur type*, or of *generalized Schur type* associated with the give module (sheaf), using the so-called *Schur* functors. Then using global method we construct canonical homomorphisms between these modules(sheaves). We will get canonical isomorphisms if the original sheaf is locally free using idea of algebraic geometry, which is in fact a generalization of result in linear algebra. In the final section, we give canonical complexes using homomorphisms constructed before, and these complexes will become split exact sequence if further condition holds. And we could use local method to give concrete descriptions of these canonical homomorphisms.

## Keywords:

Tensor Sheaves of *Schur* Types, Generalized isomorphism on determinant, Canonical Homomorphisms, Canonical Split Sequences.

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## 1 Introduction

Tensor Power, Symmetric Power and Wedge Power are basic tools used in mathematics. We are going to list its applications in linear algebra, representation theory and algebraic geometry.

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\*Department of Mathematics, Capital Normal University, Beijing 100048,China.  
Email:jkchen003@gmail.com Tel:+86-13811897120

For linear algebra, let  $V$  be a  $n$  dimensional vector space over  $k$ , and  $f \in \text{End}_k(V)$ , then using functorial property of tensor product, we may get the following homomorphism  $T_k^r(f) : T_k^r(V) \rightarrow T_k^r(V)$ . Similar homomorphisms can also be induced for symmetric power and wedge power. Then there exists the following equality  $\det(T_k^r(f)) = (\det(f))^{rm^{r-1}}$ ,  $\det(S_k^r(f)) = (\det(f))^{\frac{(n+r-1)!}{n!(r-1)!}}$  and  $\det(\wedge_k^r(f)) = (\det(f))^{\frac{(n-1)!}{(r-1)!(n-r)!}}$ . A special case of the statements before is the following equality:

$$\begin{vmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & b^2c + 2abd & b^2d \\ ac^2 & bc^2 + 2acd & ad^2 + 2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^6$$

where  $a, b, c, d$  are elements of commutative ring  $R$ , see [5] or [6] for details. In these references, the proof is an algebraic proof, in this place we are going to prove similar results into a more general fact using geometric method.

Next we are going to state some applications of these operations in representation theory. For a given representation, we may use these operations to construct new representations, such as the basic example that if  $V$  is a finite dimensional complex vector space, then we have a canonical  $GL(V)$ -invariant decomposition  $V \otimes V \cong \text{Sym}^2(V) \oplus \wedge^2(V)$ , and more generally, we may write  $V \otimes V \otimes V \cong \text{Sym}^3(V) \oplus \wedge^3(V) \oplus$  another spaces, see [3] or [10] for details. Schur functors are generalized symmetric powers and wedge powers used in construction of new representations. As a general concrete example, there admits the following decomposition,

$\text{Sym}^n(\text{Sym}^2 V) \cong \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \text{Sym}^{2n-4i} V$  One basic fact for complex representation theory is that  $\mathbb{C}$  is an algebraic closed field, so if we want to extend these results into general base, similar results may NOT globally hold, which means these properties (isomorphisms, or equalities) hold if we add some localization property.

Finally these operations occur in algebraic geometry, such as considering the Grassmanian space  $\mathbb{G}(n, m)$ , which means the set of  $m$  dimensional subspaces of a given  $n$  dimensional vector space. The standard procedure is taking  $r$ -th wedge of  $V$ , we denote this space by  $\wedge^r(V)$ . In this method, every  $m$  dimensional subspace becomes 1 dimensional vector space, then as well known, the set of these 1 dimensional subspaces becomes a geometric object, namely  $P(\wedge^r(V))$ , or equivalently  $\text{Proj}(S(\wedge^r(V)))$ . In this construction, symmetric power and wedge power occurs naturally.

In this paper, we will define tensor modules (resp. sheaves) of *Schur type*, and more generally, *generalized Schur type* associated with general  $R$ -modules (resp.  $\mathcal{O}_X$ -modules), and investigate canonical homomorphisms between these kind of modules. In section 2, using the idea of algebraic geometry, we will give a geometric global canonical isomorphism of such kind of modules, if the origi-

nal module is locally free. In section 3, we will study several kinds of canonical homomorphisms of tensor sheaves. Note that for general tensor modules of *Schur type*, we can not use local coordinates as for the locally free case. The reason is for locally free sheaves, the gluing data information is reflected by a  $n \times n$  invertible matrix, but for general sheaves, we know nothing about the gluing data information. The main idea for our method is that tensor sheaves of *Schur type* admit canonical permutation group's action. In fact, this method is already used in Prof. ke-zheng Li's lectures on de Rham Complexes. For details, see [7].

In the final section, we will give two concrete examples of general theorems, in which case that these canonical homomorphisms will make complexes. And under some further assumption, these complexes will become split exact sequences. We may use permutations to write these canonical homomorphisms clearly, and we will use local method (locally defined homomorphisms, then check these homomorphisms are independent of the given basis), and check that it is the same with global method.

## 2 SOME PREPARATIONS

Let  $R$  be a commutative ring with 1, and  $M$  be an  $R$ -module. Using tensor product one can construct  $R$ -modules with  $M$  in the following ways:

- i) symmetric product (  $S_R^n(M)$  for any  $n \in \mathbb{N}$  );
- ii) wedge product (  $\wedge_R^n(M)$  for any  $n \in \mathbb{N}$  );
- iii) tensor product over  $R$ .

One can use the above constructions repeatedly, for a finite number of (but arbitrarily many) times. For example,

$$\mathfrak{S}(M) = S_R^r(S_R^n(M) \otimes_R \wedge_R^m(M))^{\otimes_{R^2}} \quad (1)$$

Such a construction procedure is called a *Shur type*. In many cases (e.g in algebraic geometry or representation theory) such kind of complex construction procedures of modules appear.

Note that any Shur type is canonical, i.e. for any Shur type  $\mathfrak{S}$ , any  $R$ -module homomorphism  $f : M \rightarrow N$  induces an  $R$ -module homomorphism  $\mathfrak{S}(f) : \mathfrak{S}(M) \rightarrow \mathfrak{S}(N)$  canonically.

For any Shur type  $\mathfrak{S}$ , the construction procedure gives an epimorphism  $M^{\otimes_{R^d}} \rightarrow \mathfrak{S}(M)$ . We call  $d$  the *degree* of  $\mathfrak{S}$ , denoted by  $d(\mathfrak{S})$ . For example, it is not hard to see the Shur type in (1) has  $d(\mathfrak{S}) = 2r(n + m)$ .

In some cases one also uses the following way of construction in addition to i-iii):

- iv) homomorphism module  $Hom_R(\cdot, \cdot)$ ,

especially for locally free modules of finite rank. For example,

$$\mathfrak{S}'(M) = S_R^r(S_R^n(M) \otimes_R \text{Hom}_R(\wedge_R^m(M), M^{\otimes_R 2})) \quad (2)$$

Such a construction procedure is called a *generalized Shur type*.

Let  $X$  be a scheme and  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . Then any Shur type  $\mathfrak{S}$  defines a quasi-coherent sheaf  $\mathfrak{S}(\mathcal{F})$ , because  $\mathfrak{S}$  is canonical. For example, if the Shur type  $\mathfrak{S}$  is as in (1), then

$$\mathfrak{S}(\mathcal{F}) = S_{\mathcal{O}_X}^r(S_{\mathcal{O}_X}^n(\mathcal{F}) \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^m(\mathcal{F}))^{\otimes_{\mathcal{O}_X} 2}$$

Furthermore, if  $f : \mathcal{F} \rightarrow \mathcal{F}'$  is a homomorphism of quasi-coherent sheaves on  $X$ , then  $f$  induces a homomorphism  $\mathfrak{S}(f) : \mathfrak{S}(\mathcal{F}) \rightarrow \mathfrak{S}(\mathcal{F}')$  canonically.

If  $\mathfrak{S}'$  is a generalized Shur type and  $\mathcal{F}$  is locally free of finite rank, one can also define  $\mathfrak{S}'(\mathcal{F})$ . For example, if the Shur type  $\mathfrak{S}'$  is as in (2), then

$$\mathfrak{S}'(\mathcal{F}) = S_{\mathcal{O}_X}^r(S_{\mathcal{O}_X}^n(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\wedge_{\mathcal{O}_X}^m(\mathcal{F}), \mathcal{F}^{\otimes_{\mathcal{O}_X} 2}))$$

**Definition 2.1.** Let  $R$  be a commutative ring with 1, and  $M$  be an  $R$ -module. By a *tensor module* (resp. *generalized tensor module*) of  $M$  over  $R$  we mean a direct sum  $\mathfrak{S}(M) = \mathfrak{S}_1(M) \oplus \cdots \oplus \mathfrak{S}_n(M)$  (resp.  $\mathfrak{S}'(M) = \mathfrak{S}'_1(M) \oplus \cdots \oplus \mathfrak{S}'_n(M)$ ), for some Shur types  $\mathfrak{S}_1, \dots, \mathfrak{S}_n$  (resp. generalized Shur types  $\mathfrak{S}'_1, \dots, \mathfrak{S}'_n$ ).

Let  $X$  be a scheme and  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . By a *tensor sheaf* of  $\mathcal{F}$  we mean a direct sum  $\mathfrak{S}(\mathcal{F}) = \mathfrak{S}_1(\mathcal{F}) \oplus \cdots \oplus \mathfrak{S}_n(\mathcal{F})$ , for some Shur types  $\mathfrak{S}_1, \dots, \mathfrak{S}_n$ . Furthermore, if  $\mathcal{F}$  is locally free of finite rank, then by a *generalized tensor sheaf* of  $\mathcal{F}$  we mean a direct sum  $\mathfrak{S}'(\mathcal{F}) = \mathfrak{S}'_1(\mathcal{F}) \oplus \cdots \oplus \mathfrak{S}'_n(\mathcal{F})$ , for some generalized Shur types  $\mathfrak{S}'_1, \dots, \mathfrak{S}'_n$ .

By the above argument, we see that any  $\mathfrak{S}$  in Definition 1 gives a covariant functor  $\mathfrak{M}_R \rightarrow \mathfrak{M}_R$ , where  $\mathfrak{M}_R$  is the category of  $R$ -modules; and a covariant functor  $\mathfrak{Coh}_X \rightarrow \mathfrak{Coh}_X$ , where  $\mathfrak{Coh}_X$  is the category of quasi-coherent sheaves over  $X$ .

### 3 CANONICAL ISOMORPHISM

In this section, we are going to prove a global isomorphism of tensor sheaves of *Schur type* associated with locally free sheave  $\mathcal{F}$  of finite rank.

**Theorem 3.1.** Let  $(X, \mathcal{O}_X)$  be a scheme and  $\mathcal{F}$  be a locally free  $\mathcal{O}_X$ -module sheaf of rank  $n$ ,  $\mathcal{E}$  is a direct sum of tensor sheaves of schur type associated to  $\mathcal{F}$ ,  $\mathcal{E} =$

$\bigoplus_i \mathcal{E}_i$ , with each  $\mathcal{E}_i$  is of rank  $m_i$  and of degree  $d_i$ . Then we have the following canonical isomorphism of invertible sheaves:

$$(\det \mathcal{E}) \cong (\det \mathcal{F})^{\otimes \sum_i \frac{m_i d_i}{n}}.$$

For canonical, we mean if we set  $f : X' \rightarrow X$  be a morphism of schemes, set  $\mathcal{E}' = f^* \mathcal{E}$  and set  $\mathcal{F}' = f^* \mathcal{F}$ , then apply  $f^*$  to the above isomorphism, we get

$$(\det \mathcal{E}') \cong (\det \mathcal{F}')^{\otimes \sum_i \frac{m_i d_i}{n}}.$$

*Proof.* Our proof will be divided into several steps. Note that  $\mathcal{E} = \bigoplus_i \mathcal{E}_i$ , using the wedge functor, we have:

$$\begin{aligned} \det \mathcal{E} &\cong (\det(\bigoplus_i \mathcal{E}_i)) \cong \bigoplus_{i_1+i_2+\dots=d_1+d_2+\dots} \wedge^{i_1}(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \wedge^{i_2}(\mathcal{E}_2) \otimes \dots \\ &\cong \wedge^{m_1}(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \wedge^{m_2}(\mathcal{E}_2) \otimes_{\mathcal{O}_X} \dots \\ &\cong \det(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \det(\mathcal{E}_2) \otimes_{\mathcal{O}_X} \dots \end{aligned}$$

Hence we only need to prove for each  $i$ , the following equality holds:

$$\det \mathcal{E}_i \cong (\det \mathcal{F})^{\otimes \frac{m_i d_i}{n}}.$$

So we may reduce the question to the case where  $\mathcal{E}$  is a tensor sheaf of schur type of rank  $m$  and of degree  $d$  associated to free  $\mathcal{O}_X$ -sheaf  $\mathcal{F}$  of rank  $n$ .

setp1. First assume  $\frac{md}{n}$  is an integer, We will prove this fact in the end of this step. Then  $\det \mathcal{E}$  and  $(\det \mathcal{F})^{\otimes \frac{md}{n}}$  are all invertible sheaves, so locally checking, there always admits an isomorphism, the main problem is how to glue these local data to get a global isomorphism. Our method is to prove this isomorphism is canonical. Take an affine open neighborhood  $U = \text{Spec}(R)$  of  $X$ , then  $\mathcal{F}|_U \cong \widetilde{M}$ , where  $M$  is free  $R$ -module of rank  $n$  and  $\mathcal{E}|_U \cong \widetilde{M'}$  is a tensor module of schur type associated to  $M$ . Suppose  $M$  admits a basis  $m_i$  such that  $\varphi m_i = \lambda_i m_i$  for some  $\varphi \in GL(n, R)$  and  $\lambda_i \in R$ , then according to our definition of tensor modules of schur type, which is in fact the functorial property of these operations (symmetric products, for example),  $M'$  has a canonical basis which is induced from  $m_i$ , say  $m'_i$ , and  $\varphi$  induces an automorphism  $\varphi'$  acts on  $M'$ , hence  $\varphi' m'_i = \mu_i m'_i$  for each  $i$ , and each  $\mu_i$  is a product of  $\lambda_i$ s of length  $d$ , so the sum of these eigenvalues is  $dm$  (with multiplications). Also note that  $GL(n, R)$ 's diagonal action is also kept to  $M'$ , so all the multiplications of  $\lambda_i$  in  $\mu_i$  should be equal, which means the multiplication is just  $\frac{md}{n}$ .

step2. Using notations as above,  $S_R^r M$  has a canonical basis inherited from  $M$ , say  $m_{i_1} \otimes \dots \otimes m_{i_r}$  with  $i_1 \leq i_2 \leq \dots \leq i_r$  and  $\wedge_R^r M$  has a canonical basis inherited from  $M$ , which is  $m_{i_1} \wedge \dots \wedge m_{i_r}$  with  $i_1 < i_2 < \dots < i_r$ . According to our definitions,  $\det M'$  have a canonical basis inherited from given basis of  $M$ . Hence

we have as  $(\det M)^{\otimes_R \frac{md}{n}}$  and  $\det M'$  are free  $R$ -modules of rank 1, we may define an isomorphism from one generator to the other generator,

$$\begin{aligned} \phi : (\det M)^{\otimes_R \frac{md}{n}} &\longrightarrow \det M' \\ (m_1 \wedge \cdots \wedge m_n)^{\otimes \frac{md}{n}} &\mapsto m'_1 \wedge \cdots \wedge m'_m. \end{aligned}$$

Next we need to prove this is a canonical isomorphism, which means if we take  $\alpha \in GL(n, R)$ , then  $\alpha$  also induces an automorphism on  $M'$ , say  $\alpha'$ , then there exists the following commutative diagram:

$$\begin{array}{ccc} (\det M)^{\otimes_R \frac{md}{n}} & \xrightarrow{\phi} & \det M' \\ \downarrow (\det \alpha)^{\frac{md}{n}} & & \downarrow \det \alpha' \\ (\det M)^{\otimes_R \frac{md}{n}} & \xrightarrow{\phi} & \det M' \end{array}$$

So we only need to check  $\det \alpha' = (\det \alpha)^{\frac{md}{n}}$  for all  $\alpha \in GL(n, R)$ . Note that according to step1, if  $\alpha$  admits a basis such that  $\alpha$  is diagonal, then the equality already holds!

As an element  $\alpha = (\alpha_{ij}) \in GL(n, R)$  corresponds to an algebraic homomorphism  $(\mathbb{Z}[x_{ij}]_{1 \leq i, j \leq n})_{\det(x_{ij})} \rightarrow R$ , which is defined by  $\alpha_{ij} \mapsto x_{ij}$ . In fact this is a morphism from  $\text{Spec}(R)$  to  $\text{Spec}((\mathbb{Z}[x_{ij}]_{1 \leq i, j \leq n})_{\det(x_{ij})})$ , so in order to check the equality over  $R$ , we only need to check it over  $X = \text{Spec}((\mathbb{Z}[x_{ij}]_{1 \leq i, j \leq n})_{\det(x_{ij})})$ .

step3. As  $\det \alpha' = (\det \alpha)^{\frac{md}{n}}$  defines a closed subset of  $X$ , say  $V$ . We are going to prove it contains a nonempty open subset  $U$ , hence the taking closure we have  $\bar{U} \subset \bar{V} \subset \bar{V}$ . As  $X$  is irreducible, every nonempty open subset is dense, hence we have  $V = X$ .

Set  $U = \{\varphi | \chi_\varphi \text{ has } n \text{ different eigenvalues}\}$ , according to step1, we have  $U \subset V$ . Also note that  $\chi_\varphi$  has  $n$  different eigenvalues if and only if  $\Delta_\varphi \neq 0$ , so  $U$  is an open subset of  $X$ . Set  $R_0 = (\mathbb{Z}[x_{ij}]_{1 \leq i, j \leq n})_{\det(x_{ij})}$ ,  $K = \text{q.f.}(R_0)$  and  $\varphi' = (x_{ij})$ . Then  $\chi_{\varphi'} = |\lambda I - (x_{ij})|$  be the characteristic polynomial of  $(x_{ij})$ , and  $L/K$  be the splitting field of  $\chi_{\varphi'}$  over  $K$ , then  $\varphi'$  has  $n$  different eigenvalues  $\lambda_i \in L$  such that  $\varphi' v_i = \lambda_i v_i$  for  $v_i \in L^n$  for all  $i = 1, \dots, n$ . We may take a common denominator  $f$  such that these equalities hold in a open neighborhood of the generic point of  $R_0$ , hence  $U \neq \emptyset$ , taking closure we have  $V = X$ .

As the isomorphism is canonical, hence we may glue together to get a global isomorphism.  $\square$

## 4 Canonical Homomorphisms

In this section, we are going to study canonical homomorphisms of tensor sheaves. More precisely, we will construct canonical homomorphisms for  $\mathcal{O}_X$ -module  $\mathcal{F}$ .

In order to define homomorphisms of tensor sheaves of  $\mathcal{F}$ , note that if  $\mathcal{F}$  is locally free of rank  $n$ , we can use local generators of  $\mathcal{F}(U)$  for open subset  $U \subset X$ . The reason is that for locally free sheaves, when we take local generators, any two different choices of generators are differed by a  $n \times n$  invertible matrix, and the gluing data is also reflected by  $n \times n$  invertible matrix, so in order to check the morphism is well defined, we just need to fix a basis, and prove the morphism is independent of the basis we choose before. But for general  $\mathcal{O}_X$ -module  $\mathcal{F}$ , we don't know any information of local generators and information of gluing data. But for general tensor sheaves, it admits a canonical permutation group's action, this is the point we use in this paper. In fact the main idea is already occurred in the de Rham Complex in Prof. Ke-Zheng Li's lectures in "Moduli Space And Its Applications". For details, see [7].

Let  $k$  and  $n$  be two integers,  $(X, \mathcal{O}_X)$  be a scheme and  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module.

**Theorem 4.1.** There exists canonical homomorphism

$$\varphi_{n,k} : S_{\mathcal{O}_X}^n(S_{\mathcal{O}_X}^k \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^k(\wedge_{\mathcal{O}_X}^n \mathcal{F}).$$

if  $k$  is an even integer.

There exists canonical homomorphism

$$\phi_{k,n} : S_{\mathcal{O}_X}^k(\wedge_{\mathcal{O}_X}^n \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^n(S_{\mathcal{O}_X}^k \mathcal{F}).$$

*Proof.* As  $S_{\mathcal{O}_X}^k(\wedge_{\mathcal{O}_X}^n \mathcal{F})$  and  $S_{\mathcal{O}_X}^n(S_{\mathcal{O}_X}^k \mathcal{F})$  are all quotient sheaves of  $\mathcal{F}^{\otimes(n+k)}$ , so in order to define  $\varphi_{n,k}$  and  $\phi_{k,n}$ , we only need to define endomorphisms of  $\mathcal{F}^{\otimes(n+k)}$  such that  $\varphi_{n,k}$  and  $\phi_{k,n}$  can be induced from these endomorphisms.

Note  $\mathcal{F}^{\otimes(n+k)}$  admits a canonical permutation group  $\mathfrak{S}_{n+k}$ 's action, so in order to define global homomorphisms, we could use these permutations. Take  $\sigma_2, \dots, \sigma_n \in \mathfrak{S}_k$ , define  $\tilde{\varphi}_{n,k}$  as follows:

$$\begin{array}{ccc} \mathcal{F}^{\otimes(n+k)} & \longrightarrow & \mathcal{F}^{\otimes(n+k)} \\ (a_{11} \otimes \dots \otimes a_{1k}) \otimes \dots \otimes & \mapsto & \sum_{\sigma_2, \dots, \sigma_n \in \mathfrak{S}_k} (a_{11} \otimes a_{2\sigma_2(1)} \otimes \dots \otimes a_{n\sigma_n(1)}) \otimes \\ (a_{n1} \otimes \dots \otimes a_{nk}) & & (a_{12} \otimes a_{2\sigma_2(2)} \otimes \dots \otimes a_{n\sigma_n(2)}) \otimes \dots \\ & & \otimes (a_{1k} \otimes a_{2\sigma_2(k)} \otimes \dots \otimes a_{n\sigma_n(k)}) \end{array}$$

Or equivalently, we may write this homomorphism as

$$\otimes_{i=1}^n (\otimes_{j=1}^k a_{ij}) \mapsto \sum_{\sigma_2, \dots, \sigma_n \in \mathfrak{S}_k} \otimes_{i=1}^k (a_{1i} \otimes (\otimes_{j=2}^n a_{j\sigma_j(i)})).$$

It is not difficult to verify on ideal of definitions for these 2 tensor sheaves that  $\tilde{\varphi}_{n,k}$  induces a homomorphism from  $S_{\mathcal{O}_X}^n(S_{\mathcal{O}_X}^k \mathcal{F})$  to  $S_{\mathcal{O}_X}^k(\wedge_{\mathcal{O}_X}^n \mathcal{F})$ . We denote it

by  $\varphi_{n,k}$ . If we write this homomorphism in local coordinates, it is  $\otimes_{i=1}^n (\otimes_{j=1}^k a_{ij}) \mapsto$

$$\sum_{\sigma_2, \dots, \sigma_n \in \mathfrak{S}_k} \prod_{i=1}^k (a_{1i} \wedge (\wedge_{j=2}^n a_{j\sigma_j(i)})).$$

In order to define  $\phi_{k,n}$ , we still use the fact  $\mathcal{F}^{\otimes(n+k)}$  have permutation group's canonical action. Take  $\tau_2, \dots, \tau_k \in \mathfrak{S}_n$ , we define  $\tilde{\phi}_{k,n}$  as follows:

$$\begin{array}{ccc} \mathcal{F}^{\otimes(n+k)} & \longrightarrow & \mathcal{F}^{\otimes(n+k)} \\ (a_{11} \otimes \dots \otimes a_{1n}) \otimes \dots \otimes & \mapsto & \sum_{\tau_2, \dots, \tau_k \in \mathfrak{S}_n} (-1)^{\tau(\tau_2 \dots \tau_k)} (a_{11} \otimes a_{2\tau_2(1)} \otimes \dots \otimes a_{k\tau_k(1)}) \\ (a_{k1} \otimes \dots \otimes a_{kn}) & & \otimes (a_{12} \otimes a_{2\tau_2(2)} \otimes \dots \otimes a_{k\tau_k(2)}) \otimes \\ & & \dots \otimes (a_{1n} \otimes a_{2\tau_2(n)} \otimes \dots \otimes a_{k\tau_k(n)}) \end{array}$$

Or equivalently we may write this homomorphism as

$$\otimes_{i=1}^k (\otimes_{j=1}^n a_{ij}) \mapsto \sum_{\tau_2, \dots, \tau_k \in \mathfrak{S}_n} (-1)^{\tau(\tau_2 \dots \tau_k)} \otimes_{i=1}^n (a_{1i} \otimes (\otimes_{j=2}^k a_{j\sigma_j(i)})).$$

It is easy to verify that  $\tilde{\phi}_{k,n}$  induces a homomorphism from  $S_{\mathcal{O}_X}^k (\wedge_{\mathcal{O}_X}^n \mathcal{F})$  to  $S_{\mathcal{O}_X}^n (S_{\mathcal{O}_X}^k \mathcal{F})$ , which means we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes(k+n)} & \xrightarrow{\tilde{\phi}_{k,n}} & \mathcal{F}^{\otimes(k+n)} \\ \downarrow & & \downarrow \\ S_{\mathcal{O}_X}^k (\wedge_{\mathcal{O}_X}^n \mathcal{F}) & \xrightarrow{\quad \quad \quad} & S_{\mathcal{O}_X}^n (S_{\mathcal{O}_X}^k \mathcal{F}) \end{array}$$

We denote this homomorphism by  $\phi_{k,n}$ . If we write this homomorphism in local coordinates, it is  $\prod_{i=1}^k (\wedge_{j=1}^n a_{ij}) \mapsto \sum_{\tau_2, \dots, \tau_k \in \mathfrak{S}_n} ((-1)^{\tau_2 \dots \tau_k} \otimes_{i=1}^n (a_{1i} \otimes (\otimes_{j=2}^k a_{j\tau_j(i)})))$ .

This finishes the proof.  $\square$

But unfortunately for most cases we can not get the composition properties for  $\varphi_{n,k} \circ \phi_{k,n}$  and  $\phi_{k,n} \circ \varphi_{n,k}$ , even when  $\mathcal{F}$  is locally free. The main problem is although  $\varphi_{n,k}$  and  $\phi_{k,n}$  are globally well defined, it still has too many additional combinational terms if we consider compositions for these homomorphisms. A special case is  $\varphi_{2,k} \circ \phi_{k,2}$  when  $\mathcal{F}$  is locally free  $\mathcal{O}_X$ -module of rank 2. In this case, we have

$$\begin{aligned} \phi_{k,2}((x \wedge y)^k) &= \sum_{i=1}^k ((-1)^{k-i} \frac{(k-1)!}{(k-i)!(i-1)!} x^i \otimes y^{k-i} \cdot x^{k-i} \otimes y^i). \\ \varphi_{2,k}(x^i \otimes y^{k-i} \cdot x^{k-i} \otimes y^i) &= (-1)^{k-i} (k-i)! i! (x \wedge y)^k. \end{aligned}$$

So the composition is a scalar multiplication by  $\sum_{i=1}^k (-1)^{k-i} \frac{(k-1)!}{(k-i)!(i-1)!} (-1)^{k-i} (k-i)! i! = \sum_{i=1}^k i \cdot (k-1)! = \frac{(k+1)!}{2}$ . So we have  $\varphi_{2,k} \circ \phi_{k,2} = \frac{(k+1)!}{2} \cdot id_{S_{\mathcal{O}_X}^k (\wedge_{\mathcal{O}_X}^2 \mathcal{F})}$ . We also compute



the case  $\varphi_{3,2} \circ \phi_{2,3}$  when  $\mathcal{F}$  is locally free of rank 3, it is a scalar multiplication by 12, which is equal to  $\frac{(2+3-1)!}{2}$ . Hence we think the general case is still true. Namely the equality

$$\varphi_{n,k} \circ \phi_{k,n} = \frac{(k+n-1)!}{2} \cdot id_{S^k_{\mathcal{O}_X}(\wedge^n_{\mathcal{O}_X} \mathcal{F})}.$$

if  $\mathcal{F}$  is locally free of rank  $n$ .

Next we are going to use this theorem to get canonical homomorphisms of  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$ , where  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module. Our aim to study this kind of sheaf comes from [3], as there exists a decomposition of this kind of module. But in there, they deal with representation theory, the base is  $\mathbb{C}$ , we want to get more general result over general base  $X$  and the main idea is directed by [3].

First we will state the two steps handled in [3] if  $\mathcal{F}$  is a locally free  $\mathcal{O}_X$ -module of rank 2. There exists a canonical homomorphism from  $(\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F})$  to  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$  which makes the kernel for the upper canonical surjective homomorphism. If so we could get the short exact sequence:

$$0 \rightarrow (\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F}) \rightarrow S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) \rightarrow S^{2n}_{\mathcal{O}_X} \mathcal{F} \rightarrow 0.$$

Then the problem is whether we can construct a canonical homomorphism from  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$  to  $(\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F})$  which forms a canonical retraction for the upper exact sequence. Of course this canonical retraction should be constructed from the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes 2n} & \xrightarrow{\text{unknown}} & \mathcal{F}^{\otimes 2n} \\ \downarrow & & \downarrow \\ S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) & \xrightarrow{\quad \quad \quad} & (\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F}). \end{array}$$

If these 2 steps are correct, we may write  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$  as a direct sum by induction. Hence we may decompose the locally free sheaf  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$ .

But unfortunately what we have done now is not as above. We can construct global homomorphisms. The main problem is although these homomorphisms are globally well defined, as before they have too many combinational terms. Hence for composition they don't have good properties, even when  $\mathcal{F}$  is locally free  $\mathcal{O}_X$ -module of rank 2.

Let  $n \geq 3$  be an integer.

**Theorem 4.2.** There exists the following canonical homomorphisms

$$S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X} \mathcal{F}) \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{i} S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{q} S^{2n}_{\mathcal{O}_X} \mathcal{F}$$

and

$$S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X} \mathcal{F}) \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X} \mathcal{F}) \xleftarrow{j} S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) \xleftarrow{\varphi} S^{2n}_{\mathcal{O}_X} \mathcal{F}$$

such that  $q \circ i = 0$ ,  $q \circ \varphi$  is a scalar multiplication of  $S^2_{\mathcal{O}_X} \mathcal{F}$  by  $\frac{(2n-1) \cdots (n+1)}{(n-1)!}$ .

*Proof.* It is easy to see there exists a canonical surjective homomorphism

$$S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) \longrightarrow S^{2n}_{\mathcal{O}_X} \mathcal{F} \rightarrow 0.$$

which is induced by the identity map on  $\mathcal{F}^{\otimes 2n} \rightarrow \mathcal{F}^{\otimes 2n}$ . We denote this quotient homomorphism by  $q$ .

Next set  $N$  be a subset of the set  $\{1, 2, \dots, 2n\}$  such that  $1 \in N$  and  $|N| = n$ . Define a homomorphism  $\tilde{\varphi}$  as follows:

$$\begin{aligned} \tilde{\varphi} : \quad \mathcal{F}^{\otimes(2n)} &\longrightarrow \mathcal{F}^{\otimes(2n)} \\ a_1 \otimes \cdots \otimes a_{2n} &\mapsto \sum_N ((\otimes_{i \in N} a_i) \otimes (\otimes_{j \in N^c} a_j)). \end{aligned}$$

It is not difficult to check that  $\tilde{\varphi}$  induces a homomorphism from  $S^{2n}_{\mathcal{O}_X} \mathcal{F}$  to  $S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F})$ . Namely we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes(2n)} & \xrightarrow{\tilde{\varphi}} & \mathcal{F}^{\otimes(2n)} \\ \downarrow & & \downarrow \\ S^{2n}_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\quad \varphi \quad} & S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X} \mathcal{F}) \end{array}$$

We denote this homomorphism by  $\varphi$ .

For  $q \circ \varphi$ , note that for a general term  $\alpha \in S^{2n}_{\mathcal{O}_X} \mathcal{F}$ ,  $\varphi(\alpha)$  is just proper permutations of coordinates of  $\alpha$ , it does not have any other terms, so if mapped into  $S^{2n}_{\mathcal{O}_X} \mathcal{F}$  again, they are all the same element. Hence we only need to compute the multiplication. According to the definition of the subset  $N$ , we have  $q \circ \varphi = \frac{(2n-1) \cdots (n+1)}{(n-1)!} \cdot id_{S^{2n}_{\mathcal{O}_X} \mathcal{F}}$ .

Next we will define  $i$  and  $j$ . First note that according to the theorem before, we have canonical homomorphisms

$$\begin{aligned} \phi_{2,2} : \quad S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X} \mathcal{F}) &\longrightarrow S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X} \mathcal{F}) \\ (a_{11} \wedge a_{12}) \cdot (a_{21} \wedge a_{22}) &\mapsto \frac{(a_{11} \otimes a_{21}) \cdot (a_{12} \otimes a_{22}) - (a_{11} \otimes a_{22}) \cdot (a_{12} \otimes a_{21})}{2}. \end{aligned}$$

and

$$\begin{aligned} \varphi_{2,2} : \quad S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X} \mathcal{F}) &\longrightarrow S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X} \mathcal{F}) \\ (a_{11} \otimes a_{12}) \cdot (a_{21} \otimes a_{22}) &\mapsto \frac{(a_{11} \wedge a_{21}) \cdot (a_{12} \wedge a_{22}) + (a_{11} \wedge a_{22}) \cdot (a_{12} \wedge a_{21})}{2}. \end{aligned}$$

Where  $a_{ij}$  are local coordinates of  $\mathcal{F}$  for  $i, j = 1, 2$ .

So in order to define  $i$  and  $j$ , we only need to define homomorphisms

$$f : S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F}) \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X}\mathcal{F}) \rightarrow S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X}\mathcal{F}).$$

and

$$g : S^2_{\mathcal{O}_X}(S^n_{\mathcal{O}_X}\mathcal{F}) \rightarrow S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F}) \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X}\mathcal{F}).$$

set  $i = f \circ (\phi_{2,2} \otimes id_{S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X}\mathcal{F})})$  and  $j = (\varphi_{2,2} \otimes id_{S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X}\mathcal{F})}) \circ g$ . In order to check the composition properties, we just need to check it on local generators.

Define  $\tilde{f}$  and  $\tilde{g}$  as follows:

$$\begin{aligned} \tilde{f} : \mathcal{F}^{\otimes(2n)} &\longrightarrow \mathcal{F}^{\otimes(2n)} \\ (t_1 \otimes t_2 \otimes t_3 \otimes t_4) \otimes ((a_1 \otimes \cdots \otimes a_{n-2}) \cdot (b_1 \otimes \cdots \otimes b_{n-2})) &\mapsto (t_1 t_3 a_1 \cdots a_{n-2}) \otimes (t_2 t_4 b_1 \cdots b_{n-2}) \\ &\quad + (t_1 t_4 a_1 \cdots a_{n-2}) \otimes (t_2 t_3 b_1 \cdots b_{n-2}) \\ &\quad + (t_2 t_4 a_1 \cdots a_{n-2}) \otimes (t_1 t_3 b_1 \cdots b_{n-2}) \\ &\quad + (t_2 t_3 a_1 \cdots a_{n-2}) \otimes (t_1 t_4 b_1 \cdots b_{n-2}) \end{aligned}$$

$$\begin{aligned} \tilde{g} : \mathcal{F}^{\otimes(2n)} &\longrightarrow \mathcal{F}^{\otimes(2n)} \\ (a_1 \cdots a_n) \otimes (b_1 \cdots b_n) &\mapsto \sum_{i < j} \sum_{k < l} (a_i \otimes b_k \cdot a_j \otimes b_l + a_i \otimes b_l \cdot a_j \otimes b_k) \cdot \\ &\quad (a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_n) \otimes (b_1 \cdots \widehat{b_k} \cdots \widehat{b_l} \cdots b_n) \end{aligned}$$

It is not difficult to check on ideal of definitions that  $\tilde{f}$  and  $\tilde{g}$  induce homomorphisms  $f$  and  $g$ . In order to check the computation properties we write  $i$  and  $j$  in local coordinates. We have

$$\begin{aligned} &i((x_1 \wedge y_1 \cdot x_2 \wedge y_2) \cdot (a_1 \otimes \cdots \otimes a_{n-2} \cdot b_1 \otimes \cdots \otimes b_{n-2})) \\ &= (x_1 y_2 a_1 \cdots a_{n-2}) \otimes (x_2 y_1 b_1 \cdots b_{n-2}) + (x_2 y_1 a_1 \cdots a_{n-2}) \otimes (x_1 y_2 b_1 \cdots b_{n-2}) - \\ &\quad (x_1 x_2 a_1 \cdots a_{n-2}) \otimes (y_1 y_2 b_1 \cdots b_{n-2}) - (y_1 y_2 a_1 \cdots a_{n-2}) \otimes (x_1 x_2 b_1 \cdots b_{n-2}) \end{aligned}$$

And

$$\begin{aligned} &j((a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n)) \\ &= \sum_{i < j} \sum_{k < l} -(a_i \wedge b_l \cdot a_j \wedge b_k + a_i \wedge b_k \cdot a_j \otimes b_l) \cdot \\ &\quad (a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_n) \cdot (b_1 \cdots \widehat{b_k} \cdots \widehat{b_l} \cdots b_n) \end{aligned}$$

Hence we define all the homomorphisms. It is not difficult to see that for a general element  $\alpha \in S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X}\mathcal{F}) \otimes S^2_{\mathcal{O}_X}(S^{n-2}_{\mathcal{O}_X}\mathcal{F})$ ,  $i(\alpha)$  is just permutations of coordinates of  $\alpha$ , when mapped into  $S^{2n}_{\mathcal{O}_X}\mathcal{F}$ , they are the same element. Also note that these terms have different signs, hence  $q \circ i = 0$ .

This finishes the proof.  $\square$

## 5 Canonical Decompositions

In this final section, we are going to give 2 concrete examples of the theorems in section 2.

First we are going to prove the following lemma, which is an exercise in [4], which generalized the exercise in [6].

**Lemma 5.1.** Let  $(X, \mathcal{O}_X)$  be a scheme, suppose  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  is an  $\mathcal{O}_X$ -module's exact sequence,  $\mathcal{F}'$ ,  $\mathcal{F}$  and  $\mathcal{F}''$  are locally free of rank  $m$ ,  $m+n$  and  $n$ . Then we have

$$\wedge_{\mathcal{O}_X}^{m+n} \mathcal{F} \cong \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}'' . \quad (3)$$

*Proof.* It is easy to see that  $\wedge_{\mathcal{O}_X}^{m+n} \mathcal{F}$  and  $\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}''$  are both locally free  $\mathcal{O}_X$ -modules of rank 1. Use the functorial property of  $\wedge$  and the exact sequence of  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  we may get canonical surjective homomorphism  $\wedge_{\mathcal{O}_X}^m \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^m \mathcal{F}' \rightarrow 0$ ,  $\wedge_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^n \mathcal{F}'' \rightarrow 0$ , for the upper sequence and taking the kernel we get exact sequence

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} \xrightarrow{\alpha} \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}'' \rightarrow 0 . \quad (4)$$

From the definition it is easy to see that  $\text{Ker}(\alpha)$  is a subsheaf of  $\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}$  which are locally generated by sections of the form  $(a'_1 \wedge \cdots \wedge a'_m) \otimes (a_1 \wedge \cdots \wedge a_n)$  with  $a_i \in M'$  for some  $i = 1, \dots, n$ . Also note that there exists a canonical homomorphism from  $\wedge_{\mathcal{O}_X}^m \mathcal{F} \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^{m+n} \mathcal{F}$  by mapping  $(a'_1 \wedge \cdots \wedge a'_m, a_1 \wedge \cdots \wedge a_n)$  to  $a'_1 \wedge \cdots \wedge a'_m \wedge a_1 \wedge \cdots \wedge a_n$ , in fact one only need to compute there exists the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes_{\mathcal{O}_X}(m+n)} & \xrightarrow{\text{id}} & \mathcal{F}^{\otimes_{\mathcal{O}_X}(m+n)} \\ \downarrow & & \downarrow \\ \wedge_{\mathcal{O}_X}^m \mathcal{F} \otimes \wedge_{\mathcal{O}_X}^n \mathcal{F} & \xrightarrow{\quad \quad \quad} & \wedge_{\mathcal{O}_X}^{m+n} \mathcal{F} . \end{array}$$

Also we have a canonical homomorphism  $\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^m \mathcal{F} \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}$ . Taking composition we get

$$\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^{m+n} \mathcal{F} .$$

Denote this morphism by  $\beta$ . Taking the kernel we have

$$0 \rightarrow \text{Ker}(\beta) \rightarrow \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \wedge_{\mathcal{O}_X}^{m+n} \mathcal{F} . \quad (5)$$

From the construction it is easy to see that  $\text{Ker}(\beta)$  is subsheaf of  $\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}$  locally generated by sections of the form  $(a'_1 \wedge \cdots \wedge a'_m) \otimes (a_1 \wedge \cdots \wedge a_n)$  with

some  $a'_i = a_j$  for some pair  $i, j$ . Then it is easy to see that  $\text{Ker}(\beta)$  is a submodule of  $\text{Ker}(\alpha)$ . So we may get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker}(\beta) & \rightarrow & \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} & \xrightarrow{\beta} & (\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}) / \text{Ker}(\beta) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Ker}(\alpha) & \rightarrow & \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F} & \xrightarrow{\alpha} & \wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}'' \rightarrow 0 \end{array}$$

According the snake lemma we get a canonical surjective homomorphism from  $(\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}) / \text{Ker}(\beta)$  to  $\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}''$ . So  $(\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}) / \text{Ker}(\beta)$  is locally free  $\mathcal{O}_X$ -module of rank  $\geq 1$ . But in fact  $(\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}) / \text{Ker}(\beta)$  is a submodule of  $\wedge_{\mathcal{O}_X}^{m+n} \mathcal{F}$ , which is  $\mathcal{O}_X$ -module locally free of rank 1. Thus we have  $(\wedge_{\mathcal{O}_X}^m \mathcal{F}' \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X}^n \mathcal{F}) / \text{Ker}(\beta) = \wedge_{\mathcal{O}_X}^{m+n} \mathcal{F}$  and a surjective homomorphism from the former to the latter. Thus this is an isomorphism. This finishes the proof.  $\square$

**Remark:** In here we cannot use the formula of Theorem 1.17. As in here  $\mathcal{F}'$ ,  $\mathcal{F}$  and  $\mathcal{F}''$  are locally free  $\mathcal{O}_X$ -modules, so there doesn't exist a canonical split exact sequence.

Consider the special case where  $\mathcal{F}$  is a locally free  $\mathcal{O}_X$ -module of rank 2, we have the following theorem.

**Theorem 5.2.** There exists a canonical exact sequence:

$$0 \longrightarrow S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^4 \mathcal{F} \longrightarrow 0.$$

There also admits a canonical homomorphism  $\tau : S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F})$ , composite  $\tau$  with the canonical inclusion, it is multiplication by 3. So if  $X$  is an  $\text{Spec}(\mathbb{Z}[\frac{1}{3}])$  scheme, which means if 3 is an invertible element in  $\mathcal{O}_X$ , the canonical exact sequence splits, hence we have canonical direct sum

$$S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \cong S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \oplus S_{\mathcal{O}_X}^4 \mathcal{F}.$$

*Proof.* As in this case  $\mathcal{F}$  is locally free of rank 2, we are going to prove this theorem on local constructions.

First it's easy to compute  $S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F})$ ,  $S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F})$  and  $S_{\mathcal{O}_X}^4 \mathcal{F}$  are free  $\mathcal{O}_X$ -modules of rank 1, 6 and 5. Also we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes_{\mathcal{O}_X} 4} & \xrightarrow{id} & \mathcal{F}^{\otimes_{\mathcal{O}_X} 4} \\ \downarrow & & \downarrow \\ S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) & \dashrightarrow & S_{\mathcal{O}_X}^4 \mathcal{F}. \end{array}$$

Hence there exists a canonical homomorphism  $f : S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \longrightarrow S_{\mathcal{O}_X}^4 \mathcal{F}$  which is defined by mapping local sections of the form  $m_1 \otimes m_2 \otimes m_3 \otimes m_4 \in S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F})$  to

its image in  $S^4_{\mathcal{O}_X} \mathcal{F}$ , whose kernel  $\text{Ker}(f)$  is free of rank 1, and we may get the short exact sequence

$$0 \longrightarrow \text{Ker}(f) \longrightarrow S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X} \mathcal{F}) \longrightarrow S^4_{\mathcal{O}_X} \mathcal{F} \longrightarrow 0.$$

Using the upper remark and lemma, we have

$$\wedge^6_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X} \mathcal{F})) \cong (\wedge^1_{\mathcal{O}_X} \text{Ker}(f)) \otimes_{\mathcal{O}_X} \wedge^5_{\mathcal{O}_X}(S^4_{\mathcal{O}_X} \mathcal{F}) \cong \text{Ker}(f) \otimes_{\mathcal{O}_X} \wedge^5_{\mathcal{O}_X}(S^4_{\mathcal{O}_X} \mathcal{F})$$

and

$$(\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \cong \text{Ker}(f) \otimes_{\mathcal{O}_X} (\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2}$$

Hence we may get  $\text{Ker}(f) \cong (\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2} \cong S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X} \mathcal{F})$  and the exact sequence as in the theorem.

Suppose  $x$  and  $y$  form a local basis for  $\mathcal{F}|_U = \widetilde{M}$  on some affine open neighborhood of  $U \subset X$ , it is easy to check that  $\text{Ker}(f)|_U$  is generated by the local sections of the form  $(x^{\otimes 2} \otimes y^{\otimes 2} - x \otimes y \otimes x \otimes y)$ , the inclusion of  $(\wedge^2_R M)^{\otimes 2} \longrightarrow S^2_R(S^2_R M)$  is defined by sending  $(x \wedge y) \otimes (x \wedge y)$  to  $(x^{\otimes 2} \otimes y^{\otimes 2} - x \otimes y \otimes x \otimes y)$ . Next we need to check this inclusion is independent of the choice of basis, then we may glue together to get a canonical global homomorphism. More precisely, if  $x'$  and  $y'$  form another basis of  $M$ , then the gluing data is reflected by a  $2 \times 2$  invertible matrix, say  $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ , where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R)$ . We need to check there exists the following equality:

$$\begin{aligned} (\det(f))^2(x^{\otimes 2} \otimes y^{\otimes 2} - x \otimes y \otimes x \otimes y) &= (x'^{\otimes 2} \otimes y'^{\otimes 2} - x' \otimes y' \otimes x' \otimes y') \\ &= ((ax + by)^{\otimes 2} \otimes (cx + dy)^{\otimes 2} - \\ &\quad (ax + by) \otimes (cx + dy) \otimes (ax + by) \otimes \\ &\quad (cx + dy)). \end{aligned}$$

This can be proved by direct computation. Hence we may glue together this homomorphism to get the exact sequence as described in the theorem.

Our next aim is to construct "retraction" of the canonical inclusion, which means we are going to construct a global homomorphism from  $S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X} \mathcal{F})$  to  $(\wedge^2_{\mathcal{O}_X} \mathcal{F})^{\otimes 2}$ . Note that they are all quotient sheaf of  $\mathcal{F}^{\otimes 4}$ , so we may use this original tensor sheaf  $\mathcal{F}^{\otimes 4}$  to induce the retraction. As  $\mathcal{F}^{\otimes 4}$  admits permutation group  $S_4$ 's canonical action, we can define a homomorphism  $\phi$  of  $\mathcal{F}^{\otimes 4}$  as follows

$$\begin{aligned} \phi : \quad \mathcal{F}^{\otimes 4} &\longrightarrow \mathcal{F}^{\otimes 4} \\ f_1 \otimes f_2 \otimes f_3 \otimes f_4 &\mapsto \sigma_{(23)}(f_1 \otimes f_2 \otimes f_3 \otimes f_4) \\ &\quad + \sigma_{(234)}(f_1 \otimes f_2 \otimes f_3 \otimes f_4) \end{aligned}$$

Where  $f_i$  are local sections of  $\mathcal{F}$  and  $\sigma \in S_4$  permute the coordinates of  $\mathcal{F}^{\otimes 4}$ . Next we need to check the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^{\otimes 4} & \xrightarrow{\phi} & \mathcal{F}^{\otimes 4} \\ \downarrow & & \downarrow \\ S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F}) & \xrightarrow{\quad \quad \quad} & S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X}\mathcal{F}) \end{array}$$

By a direct computation on local generators we know this is a commutative diagram. Hence we get well defined homomorphism from  $S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F})$  to  $(\wedge^2_{\mathcal{O}_X}\mathcal{F})^{\otimes 2}$ , which we denote it by  $\tau$ .

As these homomorphisms are globally well defined, so in order to check the composition property, we could fix a given basis, as  $x$  and  $y$  we give before. In this case, we could give concrete forms of  $\tau$ . By a direct computation on local generators of these sheaves, we have  $\tau \circ i = 3$  as

$$\begin{aligned} \tau(x^{\otimes 2} \otimes y^{\otimes 2} - x \otimes y \otimes x \otimes y) &= x \wedge y \otimes x \wedge y + x \wedge y \otimes x \wedge y \\ &\quad - x \wedge x \otimes y \wedge y - x \wedge y \otimes y \wedge x \\ &= 3x \wedge y \otimes x \wedge y. \end{aligned}$$

So if 3 is an invertible element in  $\mathcal{O}_X$ , which means if  $X$  is an  $\text{Spec}(\mathbb{Z}[\frac{1}{3}])$  scheme, then the inclusion  $i$  has retraction by making  $\tau' = \frac{1}{3}\tau$  and the sequence is a natural split exact sequence. This finishes the proof.  $\square$

**Remark:** Compare this theorem with theorem2.1, one can check these 2 ways, globally define by permutation group's action and locally define by local generators then glue together, are the same.

Before state the following example, we introduce some notations. Note that  $\mathcal{F}^{\otimes 4}$  admits  $S_4$ 's permutation action by permute  $\mathcal{F}^{\otimes 4}$ 's coordinates, and this action is canonical. Set  $\sigma = (12) \in S_4$ , we will denote the canonical automorphism of  $\mathcal{F}^{\otimes 4}$  by permutating the 1st and 2nd coordinate by  $\sigma_{(12)}$ . The identity element's action on  $\mathcal{F}^{\otimes 4}$  will be denoted by  $\sigma_{id}$ .

Define the following homomorphism of  $\mathcal{F}^{\otimes 4}$  as follows:

$$\mathcal{F}^{\otimes 4} \xrightarrow{f_1} \mathcal{F}^{\otimes 4} \xrightarrow{f_2} \mathcal{F}^{\otimes 4} \xrightarrow{f_3} \mathcal{F}^{\otimes 4}$$

where  $f_1 = \sigma_{id} - \sigma_{(23)} + \sigma_{(234)}$ ,  $f_2 = \sigma_{(23)} - \sigma_{(234)}$  and  $f_3$  is identity map. By a direct computation, we may get the following homomorphism of tensor sheaves:

$$\wedge^4_{\mathcal{O}_X}\mathcal{F} \xrightarrow{\alpha_1} S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X}\mathcal{F}) \xrightarrow{\alpha_2} S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F}) \xrightarrow{\alpha_3} S^4_{\mathcal{O}_X}\mathcal{F}.$$

We denote the induced morphism by  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ .

We can also define homomorphisms of  $\mathcal{F}^{\otimes 4}$  as follows:

$$\mathcal{F}^{\otimes 4} \xrightarrow{g_3} \mathcal{F}^{\otimes 4} \xrightarrow{g_2} \mathcal{F}^{\otimes 4} \xrightarrow{g_1} \mathcal{F}^{\otimes 4}$$

where  $g_3 = \sigma_{id} + \sigma_{(23)} + \sigma_{(234)}$ ,  $g_2 = \sigma_{(23)} + \sigma_{(234)}$  and  $f_3$  is identity map. By a direct computation, we may get the following homomorphism of tensor sheaves:

$$S_{\mathcal{O}_X}^4 \mathcal{F} \xrightarrow{\beta_3} S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\beta_2} S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\beta_1} \wedge_{\mathcal{O}_X}^4 \mathcal{F}.$$

We denote the induced homomorphism of tensor sheaves by  $\beta_3, \beta_2$  and  $\beta_1$ .

Next we are going to prove the following theorem.

**Theorem 5.3.** There exists a canonical  $\mathcal{O}_X$ -module complex:

$$0 \rightarrow \wedge_{\mathcal{O}_X}^4 \mathcal{F} \xrightarrow{\alpha_1} S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\alpha_2} S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\alpha_3} S_{\mathcal{O}_X}^4 \mathcal{F} \rightarrow 0. \quad (6)$$

There also exists a canonical  $\mathcal{O}_X$ -module complex:

$$0 \rightarrow S_{\mathcal{O}_X}^4 \mathcal{F} \xrightarrow{\beta_3} S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\beta_2} S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \xrightarrow{\beta_1} \wedge_{\mathcal{O}_X}^4 \mathcal{F} \rightarrow 0 \quad (7)$$

We have  $\beta_1 \circ \alpha_1$  and  $\alpha_3 \circ \beta_3$  are equal to multiplication by 3. Furthermore if  $X$  is an  $\text{Spec}(\mathbb{Z}[\frac{1}{3}])$  scheme, which means if 3 is an invertible element in  $\mathcal{O}_X$ , the upper 2 complexes are both split exact sequences. Hence there exists a canonical isomorphism  $\wedge_{\mathcal{O}_X}^4 \mathcal{F} \oplus S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2 \mathcal{F}) \cong S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2 \mathcal{F}) \oplus S_{\mathcal{O}_X}^4 \mathcal{F}$ <sup>1</sup>.

*Proof.* First we will check the complex properties of (6) and (7). Note that all  $\alpha_i$  and  $\beta_i$  are induced from homomorphisms of  $\mathcal{F}^{\otimes 4}$ , so in order to check the complex property of (6) and (7). We don't have to check on  $\alpha_i$  or  $\beta_i$ , we just need to compute the composition of  $f$  or  $g$ , and prove the image in the corresponding sheaf is trivial. Take  $\alpha_2 \circ \alpha_1$  as example. We have

$$\begin{aligned} f_2 \circ f_1 &= f_2(\sigma_{id} - \sigma_{(23)} + \sigma_{(234)}) \\ &= (\sigma_{(23)} - \sigma_{(234)}) - (\sigma_{id} - \sigma_{(24)}) + (\sigma_{(34)} - \sigma_{(243)}) \end{aligned}$$

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<sup>1</sup>If  $\mathcal{F}$  is free modules, then it can be computed by hand that the upper modules are of the same rank although it looks complicated. Namely we need to prove

$$\binom{n}{4} + \left( \binom{n+1}{2} + 1 \right) = \left( \binom{n}{2} + 1 \right) + \binom{n+3}{4}.$$

This can be proved by easy computation.



Consider its image in the sheaf  $S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2\mathcal{F})$ , it is easy to check it is 0. We may also compute  $\beta_2 \circ \beta_2$ . As

$$\begin{aligned} g_2 \circ g_3 &= (\sigma_{(23)} + \sigma_{(234)}) \circ (\sigma_{id} + \sigma_{(23)} + \sigma_{(234)}) \\ &= (\sigma_{(23)} + \sigma_{(234)}) + (\sigma_{id} + \sigma_{(24)}) + (\sigma_{(34)} + \sigma_{(243)}) \end{aligned}$$

Consider its image in the sheaf  $S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2\mathcal{F})$ , we also have  $\beta_2 \circ \beta_3 = 0$ . As  $f_3$  and  $g_1$  are induced from identity, the left can be proved by easy computation.

Next we need to check the composition properties of these homomorphisms. Take  $\beta_1 \circ \alpha_1$  as an example. We have

$$\begin{aligned} g_1 \circ f_1 &= \sigma_{id} \circ (\sigma_{id} - \sigma_{(23)} + \sigma_{(234)}) \\ &= \sigma_{id} - \sigma_{(23)} + \sigma_{(234)}. \end{aligned}$$

Note that in  $\wedge_{\mathcal{O}_X}^4\mathcal{F}$ , all these terms is equal to  $\sigma_{id}$ 's action, hence the composition is multiply by 3. Similar result can be proved for  $\alpha_3 \circ \beta_3$ , as

$$\begin{aligned} f_3 \circ g_3 &= \sigma_{id} \circ (\sigma_{id} + \sigma_{(23)} + \sigma_{(234)}) \\ &= \sigma_{id} + \sigma_{(23)} + \sigma_{(234)} \end{aligned}$$

Note that the image is in the sheaf  $S_{\mathcal{O}_X}^4\mathcal{F}$ , hence they are all equal to  $\sigma_{id}$ 's action, which means the composition is multiply by 3.

When  $X$  is an  $\text{Spec}(\mathbb{Z}[\frac{1}{3}])$  scheme, which means if 3 is an invertible element in  $\mathcal{O}_X$ , we have  $\alpha_1$  and  $\beta_3$  are injective,  $\alpha_3$  and  $\beta_1$  are surjective. This proves the exactness of (6) and (7) at the point of  $\wedge_{\mathcal{O}_X}^4\mathcal{F}$  and  $S_{\mathcal{O}_X}^4\mathcal{F}$ . As

$$\begin{aligned} g_2 \circ f_2 &= (\sigma_{(23)} + \sigma_{(234)}) \circ (\sigma_{(23)} - \sigma_{(234)}) \\ &= (\sigma_{id} - \sigma_{(34)}) + (\sigma_{(24)} - \sigma_{(243)}) \end{aligned}$$

If we consider its image in the sheaf  $S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2\mathcal{F})$ , we have2

$$\begin{aligned} &(\sigma_{id} - \sigma_{(34)}) + (\sigma_{(24)} - \sigma_{(243)}) \\ &= 3\sigma_{id} - (\sigma_{id} - \sigma_{(24)} + \sigma_{(243)}) \end{aligned}$$

Which is just  $3\sigma_{id} - \alpha_1$ , hence we get  $\beta_2 \circ \alpha_2 = 3id - \alpha_1$ . Hence for any local section  $f$  of the subsheaf  $\text{Ker}(\alpha_2)$ , we have local section  $f' \in \wedge_{\mathcal{O}_X}^4\mathcal{F}$  such that  $3f - \alpha_1(f') = 0$ , namely we have  $f = \frac{1}{3}\alpha_1(f')$ . This means  $\text{Ker}(\alpha_2) = \text{im}(\alpha_1)$ . So the exactness at  $S_{\mathcal{O}_X}^2(\wedge_{\mathcal{O}_X}^2\mathcal{F})$  in the complex (6) is exact. Similar result can be proved for exactness at  $S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2\mathcal{F})$  for the complex (7). At last we need to check the exactness property at  $S_{\mathcal{O}_X}^2(S_{\mathcal{O}_X}^2\mathcal{F})$  in the complex (6). As

$$g_3 \circ f_3 = (\sigma_{id} + \sigma_{(23)} + \sigma_{(234)}) \circ \sigma_{id} = \sigma_{id} + \sigma_{(23)} + \sigma_{(234)}.$$

Consider its image in the sheaf  $S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F})$ , we have the image is equal to  $\sigma_{id} + \sigma_{(23)} + \sigma_{(24)}$ , also note that

$$\begin{aligned}\alpha_2 &= \sigma_{(23)} - \sigma_{(234)} \\ \alpha_2 \circ \sigma_{(24)} &= \sigma_{(243)} - \sigma_{(34)} \\ \alpha_2 \circ \sigma_{(243)} &= \sigma_{(24)} - \sigma_{id}\end{aligned}$$

So in the sheaf  $S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F})$ , there images are equal to  $\sigma_{(23)} - \sigma_{(24)}$ ,  $\sigma_{(23)} - \sigma_{id}$  and  $\sigma_{(24)} - \sigma_{id}$ . Hence we have  $\beta_3 \circ \alpha_3 = 3\sigma_{id} + \alpha_2(\sigma_{(24)} + \sigma_{(243)})$ . As  $\sigma_{(24)}$  and  $\sigma_{(243)}$  can be regarded as automorphisms, hence we may write  $\beta_3 \circ \alpha_3 = 3\sigma_{id} + \alpha_2$ , so the exactness of (6) at  $S^2_{\mathcal{O}_X}(S^2_{\mathcal{O}_X}\mathcal{F})$  is proved. Similar result can be proved at  $S^2_{\mathcal{O}_X}(\wedge^2_{\mathcal{O}_X}\mathcal{F})$  in the complex (7). This ends the proof.  $\square$

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